On the Existence of Pure Strategy Nash Equilibria in Two Person Discrete Games

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Abstract

We construct a generalized two-person discrete strategy static game of complete information where continuity, convexity and compactness cannot be invoked to show the existence of Pure Strategy Nash Equilibrium. We show that, when Best Responses are unique from both sides, a condition of Minimal Acyclicity is necessary and sufficient for the existence of Pure Strategy Nash Equilibria.

Key Words: Pure Strategy Nash Equilibrium; Best Response; Minimal Acyclicity

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Section 1. Introduction

This paper investigates the sufficient and necessary conditions for the existence of pure strategy Nash Equilibria (PSNE) in static games of complete information. It is not always possible to find PSNE. As in the Matching Pennies Game or the Inspection Game, the only solution is in terms of mixed strategies. This seems problematic to those theorists who find mixed strategies hard to interpret and justify.

Nash (1950) proved the existence of mixed strategy Nash equilibrium implicitly conceding the point that PSNE may not always exist. Subsequent research has searched for sufficient and necessary conditions for the existence of PSNE motivated by the concerns that mixed strategies are hard to interpret and justify. Different notions of continuity, convexity and applicable fixed point theorems along with some restrictions on payoff functions have been used to demonstrate the existence of PSNE. Debreu (1952), and Glicksberg (1952) have shown that in a strategic-form game, if the strategy spaces are non-empty, convex and compact subsets of the Euclidean space and if the payoff functions with respect to strategy profiles are continuous and with respect to own strategies are quasi-concave, then there exists a PSNE. Some other works are also important and worthy of mention. These include those of Schmeidler (1973), Mas-Colell (1984) and Khan and Sun (1995). Schmeidler (1973) showed that in a special class of games – in which payoffs to each player depends on his choice and the average choice of others, a PSNE exists in every such game with a continuum of players. Khan and Sun (1995) highlighted the importance of denumerability in such games to guarantee the existence of PSNE. Mas-Colell (1984) proved the equilibrium distribution for non-atomic games. Carmona (2006) considers an asymptotic version of Mas-Colell's theorem and shows that if the players' payoff functions are selected from an equi-continuous family, then all sufficiently large games have a PSNE. Ziad (1999) considers the class of the non-zero-sum two person games on the unit square and shows that if the payoff functions are continuous and satisfies a weaker condition than concavity or quasi-concavity then these games possess pure strategy Nash equilibrium. Lu (2007) proves a result ensuring the existence of pure strategy nash equilibrium by applying the unity partition theorem and a fixed point theorem. Recently, focus has shifted to proving the existence of PSNE with a large class of discontinuities. Dasgupta and Maskin (1986) showed that the existence of PSNE was compatible with discontinuous payoff functions as long as quasi-concavity of the payoff function in own strategy of a player was guaranteed and upper semi-continuity of the payoff function in the strategy profile replaced continuity and the strategy set of each player was nonempty, convex and compact subset of a finite dimensional Euclidean space, for all players. Sion (1958) proved the existence of PSNE in two-person zero-sum games. Reny (1999) demonstrates the existence of pure strategy Nash equilibrium for a large class of discontinuous games generalizing the mixed strategy equilibrium results of Nash (1950), Dasgupta and Maskin (1986) and others by virtue of the assumptions that strategy spaces are convex and compact, payoffs are quasi-concave in the owner's strategy and the game is better reply secure. Athey (1997) has shown that a PSNE exists in a game of incomplete information when the following condition holds: whenever each opponent uses a non-decreasing strategy, a player's best response is also non-decreasing (the Single Crossing Condition). Future research can shed light on the use of this assumption. It should also be noted that Gale (1953) has shown a convergence process in

N-person games with prefect information such that players without playing dominated strategies and playing equivalent strategies with equal probabilities, the process will converge to a game where each player has exactly one pure strategy.

I propose to identify a set of necessary and sufficient conditions for the existence a PSNE in a two player discrete game (with finite strategy points and where the payoff functions are neither continuous in the strategy profile, nor quasi-concave in own strategies) under some assumptions imposed on candidates for equilibrium strategies.

I first show the essence of the problem in the matching pennies game and discuss the intuition about how to reach PSNE.

Table 1. The Matching Pennies Game

Strategies	A_1	A_1
A_1	-1,1	1,-1
A_1	1,-1	-1,1

Note that in game depicted above, equilibrium in pure strategies fails to exist due to the persistence of incentives to deviate unilaterally from any candidate solution. This is the justification for the existence of a cycle depicted by the arrows. In order for equilibrium to exist therefore, one must have a condition of minimal acyclicity.

Section 2. A Game with 2 Players and M x N strategies

Assumption 0. There exists the player 2 strategy m_{2j} $\forall$ j=1,…N and there exists the player 1 strategy m_{1i} $\forall$ i =1,… M

It is reasonable to exclude certain classes of strategies which are not candidates for Nash Equilibrium. The matrix which is derived from this kind of exclusion operation on the original matrix should satisfy the following property:

Assumption 1. Uniqueness of Best Response functions (BR)

(a) **For each m_{2j} $\exists$ exists a unique m_{1i} s.t $m_{1i} = BR(m_{2j})$ $\forall$ j=1,…N and**

(b) **for each m_{1i} $\exists$ exists a unique m_{2j} s.t $m_{2j} = BR(m_{1i})$ $\forall$ i =1,…M**

(c) **For each m_{1i} $\exists$ a unique m_{2j} s.t $m_{i1} = BR(m_{2j})$ $\forall$ i =1,…M and j=1,…N.**

(d) **For each m_{2j} $\exists$ a unique m_{1i} s.t $m_{2j} = BR(m_{1i})$ $\forall$ i =1,…M and j=1,…N.**

The above condition states that in order to find PSNE, one has to restrict the investigation by excluding those strategies which are *not best responses*. Any strategy that is not a best response will be excluded by the Assumption 1 or BR. Further, any strategy cannot be a best response to more than one strategies of the other player. The conditions (c) and (d) can be justified by the fact that if a strategy is a best response to more than one strategy, then the original strategy set can be so partitioned that if you find a surviving strategy then it is a best response to only one corresponding surviving strategy. Note that the Assumption 1 (condition BR) is necessary to find PSNE in discrete games since one cannot invoke either the convexity and compactness of the strategy sets or the continuity of the payoff functions.

To satisfy the BR condition, therefore each player also must have the same number of strategies. The implication about dimensionality is contained in the following corollary:

Corollary 1. The matrix which satisfies the condition BR is of the dimension K x K where $K \leq$ [min $(M - x , N - y)]$ where x and y are the strategies which are not best responses.

The condition BR is one of the necessary conditions for finding PSNE, but it remains to find the necessary and sufficient condition(s) for proving the existence of PSNE given condition BR. Remember that the problem lies in the cyclic nature of incentives to deviate from any candidate Pure Strategy solution.. First, let us define the nature of the problem. Let us redefine notation as $m_{1i} = a_i$ and $m_{2j} = b_j$

Definition. A q-cycle is defined as a sequence ordering within a Q x Q matrix ($Q \leq K$) such that $a_i = BR(b_j)$ and $b_j = BR(a_{i+1})$ $\forall$ $i = j \leq Q-1$ and such that $a_Q = BR(b_Q)$ and $b_Q = BR(a_1)$.

Of course, a direct confrontation of the problem would lead us to look for the following:

Minimal Acyclicity condition (MA): Let q be the length of a q-cycle and n_q be the number of q-cycles. Then the condition MA is given by the following inequality: $K > \sum_q q.n_q$. The Residual Matrix is defined by the dimension $K - \sum_q q.n_q$.

Proposition 1. Given condition BR, a pure strategy Nash Equilibrium will always exist if and only if the Minimal Acyclicity condition MA is satisfied

Proof.

(a) *Sufficiency:* Suppose that MA is satisfied and that there does not exist a pure strategy equilibrium. Consider the residual matrix. The only way that a pure strategy Nash Equilibrium can fail to exist is if there exists a monotone sequence. Order the strategies such that a_i =BR(b_j) and b_j =BR(a_{i+1}) $\forall$ i = j ≤ K-1= min(M-x,N-y) – 1 and such that a_K =BR(b_K). But b_K is not a best response to a_i $\forall$ i s.t 2 < i ≤ K-1 since the best responses are unique, and b_K is not a best response to a_1 since the residual matrix is acyclic. But then this contradicts the BR assumption. Therefore a pure strategy Nash Equilibrium exists.

(b) *Necessity:* Suppose that condition MA is not satisfied. Then there does not exist any residual acyclic matrix. By construction, there does not exist any pure strategy Nash Equilibrium. **Q.E.D**

Section 4. Conclusion

Given the condition BR, the necessary and sufficient condition for existence of PSNE is the MA condition. Effectively, this guarantees a fixed point theorem in discrete strategy spaces without resorting to continuity, convexity and compactness. This is the interesting methodology exhibited in the paper. Future research should be directed to find the existence of PSNE in discrete strategy spaces with multiple players.

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